Invariant conserved currents for gravity

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Abstract

We develop a general approach, based on the Lagrange-Noether machinery, to the definition of invariant conserved currents for gravity theories with general coordinate and local Lorentz symmetries. In this framework, every vector field ξ on spacetime generates, in any dimension n, for any Lagrangian of gravitational plus matter fields and for any (minimal or nonminimal) type of interaction, a current $\mathcal{J}[\xi]$ with the following properties: (1) the current $(n-1)$-form $\mathcal{J}[\xi]$ is constructed from the Lagrangian and the generalized field momenta, (2) it is conserved, $d\mathcal{J}[\xi] = 0$, when the field equations are satisfied, (3) $\mathcal{J}[\xi] = d\Pi[\xi]$ "on shell", (4) the current $\mathcal{J}[\xi]$, the superpotential $\Pi[\xi]$, and the charge $\mathcal{Q}[\xi] = \int \mathcal{J}[\xi]$ are invariant under diffeomorphisms and the local Lorentz group. We present a compact derivation of the Noether currents associated with diffeomorphisms and apply the general method to compute the total energy and angular momentum of exact solutions in several physically interesting gravitational models.

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I. INTRODUCTION TO NOETHER CURRENTS

Conserved currents are related to symmetries of a physical model. In gravity theories based on the covariance principle, the action is diffeomorphism-invariant. Diffeomorphisms are generated by vector fields. Hence, vector fields should give rise to conserved currents.

Let ξ be a vector field on an n-dimensional spacetime. Then, indeed, we can associate a conserved charge to it in diffeomorphism-invariant models. For example, take a symmetric energy-momentum tensor $T_j{}^i$ (which is covariantly conserved in such theories) and a *Killing* field $\xi = \xi^i\partial_i$ (that generates an isometry of the spacetime). Then $j^i = \xi^j T_j{}^i$ is a conserved current, and a corresponding charge is defined [1] as the integral $\int_S j^i\, \partial_i \rfloor \eta$ over an $(n-1)$-hypersurface S. Moreover, it is possible to construct a conserved current $(n-1)$-form for any solution of a diffeomorphism-invariant model even when ξ is not a Killing field [2]. Such a current and the corresponding charge are scalars under general coordinate transformations. The Komar charges [3] arise in this way.

The situation becomes more complicated when, besides the diffeomorphism symmetry, the gravitational model is also invariant under the local Lorentz group $SO(1, n-1)$. This is the case for the gauge gravity models [4], for supergravity, for the so called first order formulation of standard General Relativity, and, in general, for the case when *spinor* matter is present. The problem of defining conserved quantities associated with a vector field was analysed [4, 5, 6, 7, 8] for specific Lagrangians (usually, for the Hilbert-Einstein one) and for specific types of vector fields (usually, for Killing or generalized Killing ones). Moreover, some resulting conserved quantities were *not invariant* under the local Lorentz group (e.g., in [8]).

In this letter, we present a *compact derivation of the invariant conserved currents and charges for gravity models with diffeomorphism and local Lorentz symmetries*. This is done in any dimension, for any Lagrangian of the gravitational field and of (minimally or non-minimally coupled) matter, and for any vector field ξ. We then apply our general results to different interesting gravitational models.

It is worthwhile to stress that the subject of our studies are not all currents (and the corresponding charges), but only the *Noether currents*. We recall, that Noether [9] demonstrated that a symmetry of the action gives rise to a current, the divergence of which is a linear combination of the field (Euler-Lagrange) equations. She proved also the inverse

theorem: if there is a current, the divergence of which is equal to a linear combination of the field equations, then the action is invariant under a certain symmetry. In this way, there is a one-to-one correspondence between the symmetries of the action and the currents with the property mentioned. Furthermore, if the fields satisfy the Euler-Lagrange equations ("on-shell", using the standard jargon), then such a current is conserved, i.e., its divergence vanishes. It is possible then to define the corresponding charge. Because of the direct "off-shell" relation of the Noether current with the symmetry of the action (that is, with the structure of the latter), the form of such current is determined by the Lagrangian of the theory.

In addition, in any field-theoretic model there is an infinite number of *non-Noether currents*. In n dimensions, such a current is given by an arbitrary exact $(n-1)$-form, i.e., $J^{\rm nN} := dU$ where an $(n-2)$-form U can be an arbitrary function of the fields. Obviously, $dJ^{\rm nN} \equiv 0$, and this fact holds always, independently of the structure of the Lagrangian and of the field equations. We stress this point: The divergence of such a current is *not* equal to a linear combination of the field equations. Accordingly, using the inverse Noether theorem, such a current *does not correspond to any symmetry of the action.* That is the reason why we call this a non-Noether current. As a result, in contrast to the Noether currents, the structure of non-Noether currents is completely ambiguous. One needs additional assumptions in order to fix their structure, the corresponding general discussion can be found in [10], for example.

It is clear that given a Noether current J and a non-Noether current $J^{\rm nN}$, one can define a new quantity $J' := J + J^{\rm nN}$. This is conserved "on-shell". Accordingly, the new conserved charge is defined as a sum of the Noether charge and a non-Noether charge. It is important to realize that the Noether charge and non-Noether charges are conserved *separately.* If we recall, in addition, that non-Noether currents and charges are completely arbitrary and not related either to the Lagrangian of a theory or to its dynamics, then a reasonable question arises: Why at all do we need to mix the two quantities, J and $J^{\rm nN}$? One can, instead, consistently study *only* Noether currents, which are directly related to the symmetries of the action, and the structure of which is fixed by the crucial "off-shell" relation of their divergence to the combination of the Euler-Lagrange equations. And, separately, one can consistently study *only* non-Noether currents, which are not related to any symmetry and even to any Lagrangian as such, and the structure of which should be fixed by totally

different considerations. Since the study of the latter can be found elsewhere [10] (see also the recent development in [11]), here we concentrate on the *Noether currents only*.

We then demonstrate that the structure of the Noether current associated with an arbitrary diffeomorphism ξ is uniquely determined by the "off-shell" relation (7). Furthermore, we show that the structure of the corresponding superpotential $(n-2)$-form is also uniquely fixed by the "off-shell" relation (8). We stress this crucial point: Following the original Noether's derivations [9], we find a precise form of the current and superpotential for every Lagrangian, which is invariant under diffeomorphisms and the local Lorentz group, in terms of the Lagrangian and its derivatives (1). There is no any ambiguity in our construction, since from the very beginning we have put aside all the ambiguously defined non-Noether currents.

The only "ambiguity" left in our approach is related to the fact that the Lagrangian can be shifted by a total derivative (boundary) term. But this is a controlled "ambiguity" (and, hence, not an ambiguity at all): When we change the Lagrangian in this way, we automatically change its derivatives (1) with respect to the fields Φ and "velocities" $d\Phi$. As a result, we certainly obtain a new current for a transformed Lagrangian. But the form of the new current is again uniquely determined by the new (shifted by a boundary term) Lagrangian. The one-to-one correspondents between the Lagrangians and the Noether currents is thus guaranteed in this approach.

Before we go ahead with the description of the results, it seems necessary to make a comment concerning some misunderstandings about the Noether currents and charges. A typical example can be found in [10], where on page 4 the authors write the following: "... a Noether current associated to a gauge symmetry necessarily vanishes on-shell ... up to a divergence of an arbitrary superpotential". Furthermore, on page 5 they continue with: "Note that the superpotential is completely *arbitrary* (the emphasis of the authors of [10])... . This implies in particular that the Noether charge ... is *undefined* (our emphasis) because it is given by the surface integral of an arbitrary $(n-2)$-form". These statements are misleading.

To begin with, the Noether current certainly does not "vanish on-shell". It is closed on-shell. But the current J itself is nonvanishing and it defines a conserved charge $Q = \int_S J$ (when the fields satisfy certain appropriate conditions at infinity, which is usually assumed). It is worthwhile to remember that a charge Q is, as such, a *volume* integral over an $(n-1)$-

dimensional spatial hypersurface S. And in this sense, the crucial thing for the evaluation of the total charge Q is the *physics inside* this volume S, and not mathematics at the boundary ∂S (recall how a conserved electric charge is evaluated in classical electrodynamics, for example).

Furthermore, it is certainly well known (see, [10, 12], for example), that the "on-shell conserved" currents for local symmetry groups are "on-shell exact", that is, they are expressed in terms of superpotentials. We also prove this explicitly here. However, this fact does not devaluate the Noether construction in any sense. Indeed, suppose on-shell we have $J = d\Pi_0$. Let us ask: To what extent the superpotential $(n-2)$-form is arbitrary? To study this, we assume that besides Π_0 there is another form Π_1 which also satisfies $J = d\Pi_1$. Now, by a simple subtraction of the first equation from the second one, we find that the difference $(\Pi_1 - \Pi_0)$ is closed: $d(\Pi_1 - \Pi_0) = 0$. Then, using the results on the trivial cohomologies [10] we derive that $\Pi_1 = \Pi_0 + dU$ where U is an arbitrary $(n-3)$-form. Therefore, a superpotential Π for a Noether current $J = d\Pi$ of a gauge symmetry is indeed defined nonuniquely, but the corresponding ambiguity is just a shift by an exact form, $\Pi \longrightarrow \Pi + dU$. It is easy to see that this "arbitrariness" is harmless. Indeed, starting from the charge defined as a volume integral $Q = \int_S J$, after substituting $J = d\Pi$ and using the Stokes theorem, we reduce the charge to the surface integral $Q = \int_{\partial S} \Pi$. Now, what is the effect of a shift $\Pi \longrightarrow \Pi + dU$ on the value of this integral? Well, there is no effect at all, because $\int_{\partial S} dU \equiv 0$ (use Stokes theorem and "boundary of a boundary is zero"). In other words, the statement of [10] that a Noether charge is "undefined" because of the ambiguity of a superpotential is misleading and wrong. The Noether charge is *well defined* despite a certain arbitrariness in the choice of a superpotential.

Previously, we have demonstrated [13] that the Noether-Lagrange approach works perfectly well in the standard case of the Einstein(-Cartan) theory, and the computation of the total energy and angular momentum is in an agreement with results obtained by other methods (ADM mass, Hamiltonian surface integrals, and covariant phase space charges) [2, 6, 7]. Here we apply this approach to other gravitational models, such as theories in lower and higher dimension, Brans-Dicke and higher derivative models.

II. GENERAL FORMULATION

In the theories possessing local Lorentz invariance, the gravitational field is naturally described by the 1-forms of the coframe ϑ^α and the Lorentz connection $\Gamma_\alpha{}^\beta$. The orthonormal coframe determines the lengths and angles on the spacetime manifold by introducing the line element $ds^2 = g_{\alpha\beta}\,\vartheta^\alpha \otimes \vartheta^\beta$, with the n-dimensional Minkowski tensor given by $g_{\alpha\beta} := \mathrm{diag}(1, -1, \cdots, -1)$. Using the gauge-theoretic language, we will refer to the coframe and connection as the translational and rotational fields, respectively. In this framework, both the curvature 2-form, $R_\alpha{}^\beta = d\Gamma_\alpha{}^\beta + \Gamma_\gamma{}^\beta \wedge \Gamma_\alpha{}^\gamma$, and the 2-form of torsion, $T^\alpha = D\vartheta^\alpha$, are nontrivial, in general.

Let us consider a Lagrangian n-form $V^{\rm tot} = V + L$ that describes the system of coupled gravitational, $\vartheta^\alpha, \Gamma_\alpha{}^\beta$, and matter fields, ψ^A. The latter include scalars and spinors of arbitrary rank, belonging to some representation of the Lorentz group. We assume them to be 0-forms. The gravitational Lagrangian $V = V(\vartheta^\alpha, T^\alpha, R_\alpha{}^\beta)$ depends on the covariant geometrical objects: coframe, torsion and curvature. The material Lagrangian $L = L(\psi^A, D\psi^A, \vartheta^\alpha, T^\alpha, R_\alpha{}^\beta)$ is a function of the matter fields and their derivatives, but it may also depend on the coframe, torsion and curvature. By allowing this, we include the possibility of nonminimal coupling of matter to gravity (like in the Brans-Dicke theory, for example) by means of Pauli-type terms in the action. Moreover, some of the matter fields may play the role of nondynamical Lagrange multipliers imposing various constraints. For example, the zero-torsion constraint $T^\alpha = 0$ can be introduced by means of the term $\psi_\alpha \wedge T^\alpha$ with the help of a Lagrange multiplier $(n-2)$-form ψ_α. The resulting dynamical setting of theories with local Lorentz symmetry generalizes the previous studies [6, 7].

It is convenient to collect all the fields (gravitational and matter) into a single multi-component field: $\Phi^I = \left(\vartheta^\alpha, \Gamma_\alpha{}^\beta, \psi^A\right)$. The collective index I runs over the three sectors: α ("translational"), $[\alpha\beta]$ ("rotational", labeled by antisymmetrized pairs of indices), and A (matter). The derivatives of $V^{\rm tot}(\Phi^I, d\Phi^I)$ w.r.t. the generalized "velocities" and w.r.t. the fields introduce the field momenta and the "potential energy" terms by

$$H_I := -\frac{\partial V^{\rm tot}}{\partial d\Phi^I}, \qquad E_I := \frac{\partial V^{\rm tot}}{\partial \Phi^I}, \tag{1}$$

respectively. A total variation of the Lagrangian is then

$$\delta V^{\rm tot} = \delta\Phi^I \wedge \mathcal{F}_I - d(\delta\Phi^I \wedge H_I), \tag{2}$$

where we introduced the variational derivative

$$\mathcal{F}_I := \frac{\delta V^{\rm tot}}{\delta \Phi^I} = (-1)^{p(I)} DH_I + E_I. \tag{3}$$

Here $p(I)$ denotes the rank (in the exterior sense) of the corresponding sector of the collective field ($p = 1$ for the coframe and connection and $p = 0$ for the matter field). The field equations for the coupled gravitational and matter fields read $\mathcal{F}_I = 0$.

We assume that the action of the theory is invariant under diffeomorphism and local Lorentz transformations. The total *infinitesimal symmetry variation* of the collective field consists of two terms:

$$\delta\Phi^I = \varsigma\mathcal{L}_{\{\xi,\varepsilon\}}\Phi^I := \delta_{(\varsigma\xi)}\Phi^I + \delta_{(\varsigma\varepsilon)}\Phi^I. \tag{4}$$

Here ς is an infinitesimal constant parameter. The term $\delta_{(\varsigma\xi)}\Phi^I = \varsigma\ell_\xi\Phi^I$ comes from a diffeomorphism generated by a vector field ξ, where ℓ_ξ is the Lie derivative along the latter. The second term describes a local Lorentz transformation $\delta_{(\varsigma\varepsilon)}\Phi^I = \varsigma\left[\varepsilon^\alpha{}_\beta(\rho^\beta{}_\alpha)^I{}_J\,\Phi^J - (\sigma^\beta{}_\alpha)^I\,d\varepsilon^\alpha{}_\beta\right]$. Here $\rho^I{}_J$ are the Lorentz generators, and σ^I is nontrivial only in the rotational sector (connection), in which it is equal to the identity matrix. It is convenient to introduce a special notation, $\mathcal{L}_{\{\xi,\varepsilon\}}$, for the total variation, as we did in the last equality of (4).

The condition of the invariance of the theory under a general variation (4) is read off directly from (2):

$$d(\xi\rfloor V^{\rm tot}) = (\mathcal{L}_{\{\xi,\varepsilon\}}\Phi^I)\wedge\mathcal{F}_I - d\left[(\mathcal{L}_{\{\xi,\varepsilon\}}\Phi^I)\wedge H_I\right]. \tag{5}$$

Introducing the current $(n-1)$-form

$$J[\xi,\varepsilon] := \xi\rfloor V^{\rm tot} + (\mathcal{L}_{\{\xi,\varepsilon\}}\Phi^I)\wedge H_I, \tag{6}$$

we see from (5) that

$$dJ[\xi,\varepsilon] = (\mathcal{L}_{\{\xi,\varepsilon\}}\Phi^I)\wedge\mathcal{F}_I. \tag{7}$$

Hence, this current is conserved, $dJ[\xi,\varepsilon] = 0$, for any ξ and $\varepsilon^\alpha{}_\beta$, when the field equations, $\mathcal{F}_I = 0$, are satisfied.

Using (6), (4) and the Noether identities of the diffeomorphism symmetry [13], we rewrite the current as

$$J[\xi,\varepsilon] = d\Pi[\xi,\varepsilon] + \Xi^I[\xi,\varepsilon]\wedge\mathcal{F}_I. \tag{8}$$

Here we denoted $\Pi[\xi,\varepsilon] := \Xi^I[\xi,\varepsilon]\wedge H_I$ and

$$\Xi^I[\xi,\varepsilon] := \xi\rfloor\Phi^I - \varepsilon^\alpha{}_\beta(\sigma^\beta{}_\alpha)^I. \tag{9}$$

On the solutions of the field equations, the charge is an integral over an $(n-2)$-boundary:

$$Q[\xi,\varepsilon] = \int_S J[\xi,\varepsilon] = \int_{\partial S}\Pi[\xi,\varepsilon]. \tag{10}$$

The functions $\varepsilon^\alpha{}_\beta$ parametrize the *family* of conserved currents (6) and charges (10) associated with a vector field ξ. In order to select *invariant* charges, we will have to specialize to a particular choice of ε. The trivial choice $\varepsilon^\alpha{}_\beta = 0$ yields a *noninvariant* current and charge (an explicit example was obtained recently in [8]). Indeed, then $\mathcal{L}_{\{\xi,\varepsilon\}}\Phi^I = \ell_\xi\Phi^I$, and the last term in (6) is not Lorentz invariant, since the Lie derivative ℓ_ξ is not covariant under local Lorentz transformations.

The situation is improved if we make $\mathcal{L}_{\{\xi,\varepsilon\}}$ a covariant operator by an appropriate choice of ε. This is always possible to do, although not uniquely. The choice

$$\varepsilon_{\alpha\beta} = -\Theta_{\alpha\beta} := -e_{[\alpha}\rfloor\ell_\xi\vartheta_{\beta]} \tag{11}$$

is in a certain sense *minimal*. The Lie derivative of the coframe can be decomposed as $\ell_\xi\vartheta^\alpha = (S_\beta{}^\alpha + \Theta_\beta{}^\alpha)\,\vartheta^\beta$, where the symmetric $S_{\alpha\beta} = e_{(\alpha}\rfloor\ell_\xi\vartheta_{\beta)} \equiv h^i_\alpha h^j_\beta\,\ell_\xi g_{ij}/2$ and the antisymmetric $\Theta_\beta{}^\alpha$ is given by (11). We can immediately verify that $S_\beta{}^\alpha$ is a tensor under local Lorentz transformations. Then we simply move $\Theta_\beta{}^\alpha\vartheta^\beta$ to the l.h.s. and define a "generalized Lie derivative" of the coframe as $\mathcal{L}_\xi\vartheta^\alpha := \ell_\xi\vartheta^\alpha - \Theta_\beta{}^\alpha\vartheta^\beta$. By construction, it is *covariant*, and, moreover, $\mathcal{L}_\xi\vartheta^\alpha = \mathcal{L}_{\{\xi,\varepsilon=-\Theta\}}\vartheta^\alpha$. Thus, the choice (11) is minimal in the sense that it provides a covariant generalization of the Lie derivative [14] of the coframe without any additional variables and constants, using just the coframe itself. Inserting (11) into (9), we find $\Xi_\alpha{}^\beta = \xi\rfloor\Gamma_\alpha{}^\beta + \Theta_\alpha{}^\beta$.

Our formalism thus shows that for a gravity model with diffeomorphism and local Lorentz symmetries, any vector field ξ generates an *invariant current* ($\mathcal{L}_\xi := \mathcal{L}_{\{\xi,\varepsilon=-\Theta\}}$):

$$\mathcal{J}[\xi] = \xi\rfloor V^{\rm tot} + \mathcal{L}_\xi\vartheta^\alpha\wedge H_\alpha + \mathcal{L}_\xi\Gamma_\alpha{}^\beta\wedge H^\alpha{}_\beta + \mathcal{L}_\xi\psi^A H_A. \tag{12}$$

We expand here the condensed notation in order to show how the gravitational and matter fields appear in the final formulas. The definition (1) reads: $H_\alpha = -\partial V^{\rm tot}/\partial T^\alpha$, $H^\alpha{}_\beta =$

$-\partial V^{\rm tot}/\partial R_\alpha{}^\beta$, and $H_A = -\partial V^{\rm tot}/\partial D\psi^A$. The current (12) and its derivative satisfy:

$$\mathcal{J}[\xi] = d\Pi[\xi] + \xi^\alpha \mathcal{F}_\alpha + \Xi_\alpha{}^\beta \mathcal{F}^\alpha{}_\beta, \tag{13}$$

$$d\mathcal{J}[\xi] = \mathcal{L}_\xi \vartheta^\alpha \wedge \mathcal{F}_\alpha + \mathcal{L}_\xi \Gamma_\alpha{}^\beta \wedge \mathcal{F}^\alpha{}_\beta + \mathcal{L}_\xi \psi^A \mathcal{F}_A. \tag{14}$$

Here $\Pi[\xi] := \xi^\alpha H_\alpha + \Xi_\alpha{}^\beta H^\alpha{}_\beta$. For the solutions of the field equations, the *invariant charge* (10) then reads

$$\mathcal{Q}[\xi] = \int_{\partial S} \left(\xi^\alpha H_\alpha + \Xi_\alpha{}^\beta H^\alpha{}_\beta \right), \tag{15}$$

with the $(n-2)$-dimensional boundary ∂S of a spacelike $(n-1)$-hypersurface S. Conservation of this charge, i.e., that it assumes constant values when computed on different spacelike hypersurfaces (corresponding to different times) is derived, as usual, when we integrate the conservation law $dJ[\xi, \varepsilon] = 0$ over the n-volume domain with the boundary $S_1 + S_2 + T$, where S_1 and S_2 are $(n-1)$-dimensional spacelike hypersurfaces (which correspond to the arbitrary time values t_1 and t_2, respectively) and T is a timelike surface that connects them. When the fields satisfy the "no-flux" boundary conditions such that $\int_T J = 0$, the charge (15) is constant. It is an interesting question whether the "no-flux" condition might be connected to the choice of ξ as a Killing vector. We plan to study this hypothesis elsewhere.

We always assume that the fields satisfy the "no-flux" condition, the explicit form of which should be established on a case by case basis after the spacetime dimension and the model Lagrangian is specified. One can check that these boundary conditions are fulfilled for all static and stationary configurations which we consider in our subsequent computations.

We now test the general formalism in the following concrete applications: (i) Einsteinian gravity with minimal coupling (general relativity in 4, 3, and 5 dimensions), (ii) a model with nonminimal coupling (Brans-Dicke theory), (iii) a higher-derivative gravity model.

III. EINSTEIN(-CARTAN) THEORY IN ANY DIMENSION

In n-dimensional spacetime, the Hilbert-Einstein Lagrangian with cosmological constant λ reads

$$V = -\frac{1}{2\kappa_n} \left(R^{\alpha\beta} \wedge \eta_{\alpha\beta} - 2\lambda\eta \right). \tag{16}$$

For $n \geq 4$, the relativistic gravitational constant is $\kappa_n = 2(n-3)v_{n-1}G_n/c^3$, where G_n is the Newtonian constant (the dimensionality of which depends on the dimension of space)

and $v_d = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of a $(d-1)$-dimensional unit sphere. Then $\mathcal{Q}[\xi] = (1/2\kappa_n)\int_{\partial S} {}^*\Big[dk + \xi\rfloor(\vartheta^\lambda \wedge T_\lambda)\Big]$, where $k := \xi_\alpha\,\vartheta^\alpha$. This reduces to Komar's expression for spinless matter or in vacuum, since then $T^\alpha = 0$.

In 4 dimensions, we consider the Lagrangian $V' = V + \alpha_0\, d\Phi_P$ with (16) supplemented by a topological boundary term (cf. [8]) given by the 3-form

$$\Phi_P = \eta_{\alpha\beta\mu\nu}\,\Gamma^{\alpha\beta}\wedge\left(R^{\mu\nu} + \tfrac{1}{3}\,\Gamma^{\mu\lambda}\wedge\Gamma_\lambda{}^\nu\right). \tag{17}$$

The boundary term is needed to regularize the conserved quantities for asymptotically AdS configurations, which is achieved by choosing $\alpha_0 = 3/8\kappa_4\lambda$.

Let us find the invariant charge for the Kerr-AdS solution: in the spherical coordinates (t, r, θ, φ), the corresponding coframe reads [13]:

$$\vartheta^{\hat 0} = \sqrt{\frac{\Delta}{\Sigma}}\left[c dt - a\Omega\sin^2\theta\, d\varphi\right], \tag{18}$$

$$\vartheta^{\hat 1} = \sqrt{\frac{\Sigma}{\Delta}}\,dr, \quad \vartheta^{\hat 2} = \sqrt{\frac{\Sigma}{f}}\,d\theta, \tag{19}$$

$$\vartheta^{\hat 3} = \sqrt{\frac{f}{\Sigma}}\sin\theta\left[-a\, c dt + \Omega(r^2 + a^2)\, d\varphi\right]. \tag{20}$$

Here $m = G_4 M/c^2$, and the functions are defined by

$$\Delta := (r^2 + a^2)(1 - \frac{\lambda}{3}\, r^2) - 2mr, \tag{21}$$

$$\Sigma := r^2 + a^2\cos^2\theta, \tag{22}$$

$$f := 1 + \frac{\lambda}{3}\, a^2\cos^2\theta, \quad \Omega^{-1} := 1 + \frac{\lambda}{3}\, a^2. \tag{23}$$

The invariant charges for this solution are then easily computed:

$$\mathcal{Q}'[\partial_t] = \Omega M c^2, \qquad \mathcal{Q}'[\partial_\varphi] = -\Omega^2 M c a. \tag{24}$$

They coincide with the *noncovariant* charges found in [8] for a particular choice of frame. In order to see this, one must take into account that the coframe defined by (18)-(20) differs from the one used in [8] by a factor Ω of the dt component. In other words, the frame in [8] corresponds to a change of time coordinate $t = \Omega t'$, so that $\mathcal{Q}'[\partial'_t] = \Omega^2 M c^2$. One should always remember that the result of the use of the general formula (15) depends on the input, i.e., on the configuration of the fields (metric and other), on the the integration domain in the integral, and on the vector field ξ. Careful application of this formula then reproduces the same conserved charges as in [16].

Our general approach works in any dimension: We consider now the models determined by the Lagrangian (16) for $n = 3$ and $n = 5$, respectively.

3D BTZ black hole: For the uncharged BTZ solution [15], we choose the frame:

$$\vartheta^{\hat{0}} = f c\, dt, \qquad \vartheta^{\hat{1}} = f^{-1}\, dr, \qquad \vartheta^{\hat{2}} = r d\varphi - \frac{Jc}{2r}\, dt \tag{25}$$

with $f = \sqrt{(J/2r)^2 - \lambda r^2 - m}$. The invariant charge $\mathcal{Q}[\partial_t]$ formally diverges and regularization is needed. This can be performed by the relocalization, $V \to V' = V + d\Phi$, with the help of an appropriate boundary term. Explicitly, $\Phi = -\eta_{\alpha\beta}\wedge\Delta\Gamma^{\alpha\beta}/2\kappa_3$ with $\Delta\Gamma_\alpha{}^\beta := \Gamma_\alpha{}^\beta - \overline{\Gamma}_\alpha{}^\beta$. The "background" connection is chosen as a flat connection $\overline{\Gamma}_\alpha{}^\beta := \Gamma_\alpha{}^\beta\Big|_{m=J=0}$ with nontrivial components $\overline{\Gamma}^{\hat{0}\hat{1}} = -\lambda r c\, dt$, $\overline{\Gamma}^{\hat{1}\hat{2}} = -\sqrt{-\lambda}\, r\, d\varphi$. For V' we then have

$$\mathcal{Q}'[\xi] = \frac{1}{2\kappa_3}\int_{\partial S} \eta_{\alpha\beta\lambda}\, \xi^\alpha \Delta\Gamma^{\beta\lambda}. \tag{26}$$

Direct computation for the BTZ solution yields

$$\mathcal{Q}'[\partial_t] = \frac{\pi m c}{\kappa_3}, \qquad \mathcal{Q}'[\partial_\varphi] = \frac{\pi J}{\kappa_3}. \tag{27}$$

A similar but more involved derivation can be found in [6]. Note that the relocalization above cancels the rotational contribution [second term in (15)] and replaces it with a translational one [first term in (15)]. This demonstrates the convenience of the general framework in which a Lagrangian may depend on all covariant geometrical objects, including the torsion. In this example, for the boundary term $d\Phi$ above, the derivative $H'_\alpha = -\partial V'/\partial T^\alpha \neq 0$ yields a nontrivial translational field momentum despite the fact that torsion is absent, $T^\alpha = 0$, "on shell".

5D Kerr solution can be described, see for example [16], by the line element

$$\begin{aligned} ds^2 \;=\; & c^2 dt^2 - \frac{\Sigma}{\Delta}\, dr^2 - \frac{2m}{\Sigma}\left(cdt - a\sin^2\theta\, d\varphi - b\cos^2\theta\, d\psi\right)^2 \\ & - \Sigma\, d\theta^2 - (r^2 + a^2)\sin^2\theta\, d\varphi^2 - (r^2 + b^2)\cos^2\theta\, d\psi^2. \end{aligned} \tag{28}$$

Here $\Sigma = r^2 + a^2\cos^2\theta + b^2\sin^2\theta$, $\Delta = (r^2 + a^2)(r^2 + b^2)/r^2 - 2m$, with $m = G_5 M/c^2$ and $0 < t < \infty$, $0 < r < \infty$, $0 < \theta < \pi/2$, $0 < \varphi < 2\pi$, $0 < \psi < 2\pi$. We find the charges:

$$\mathcal{Q}[\partial_t] = Mc^2/2, \qquad \mathcal{Q}[\partial_\varphi] = -Mca/2, \qquad \mathcal{Q}[\partial_\psi] = -Mcb/2. \tag{29}$$

The angular momenta obtained agree with the values given in [16]. As for the total mass, its value is different from the value reported in [16], for example. However, this is the usual

"defect" of the Komar charge which is easily repaired with the help of the appropriate total derivative (boundary) term added to the Hilbert-Einstein Lagrangian, as was demonstrated in [17].

IV. BRANS-DICKE THEORY

The Brans-Dicke theory is defined by the Lagrangian

$$V = -\frac{1}{2\kappa_4}\left(\phi\,\eta_{\alpha\beta}\wedge R^{\alpha\beta} + \omega\phi^{-1}d\phi\wedge {}^*d\phi\right) - \psi_\alpha\wedge T^\alpha, \tag{30}$$

where ω is a constant and the last term imposes the vanishing torsion condition. Then

$$H_\alpha = \psi_\alpha, \qquad H_{\alpha\beta} = \frac{\phi}{2\kappa_4}\,\eta_{\alpha\beta}. \tag{31}$$

The field equation corresponding to variation w.r.t. the connection implies $\psi_\alpha = -e_\alpha\rfloor{}^*d\phi/\kappa_4$. The spherically symmetric solution [18] is given in isotropic coordinates by the line element

$$ds^2 = f^2c^2dt^2 - h^2(dr^2 + r^2d\Omega^2), \tag{32}$$

with

$$f = (1-m/r)^{1/\mu}(1+m/r)^{-1/\mu}, \qquad h = (1+m/r)^2 f^{\mu-\nu-1}, \tag{33}$$

$m = G_4M/c^2$ and the scalar field $\phi = f^\nu$. The integration constants satisfy $\mu^2 = (1+\nu)^2 - \nu(1-\omega\nu)$. Then we find

$$\mathcal{Q}[\partial_t] = Mc^2(1-\nu)/\mu. \tag{34}$$

Our result agrees with the generalized Komar construction [19] and differs from that of Hart [20].

V. HIGHER DERIVATIVE GRAVITY

Our general approach can also be applied to models with more nontrivial Lagrangians than (16) and (30). As a last example, let us now consider quadratic-curvature models in 4 dimensions. The Lagrangian 4-form of these models reads:

$$V = -\frac{1}{4\kappa_4}\,R^{\alpha\beta}\wedge{}^*\left(\sum_{I=1}^{6} b_I\,{}^{(I)}R^{\alpha\beta}\right) - \psi_\alpha\wedge T^\alpha. \tag{35}$$

The last term imposes the zero-torsion constraint. As a result, the sum in the first term contains only three of the six irreducible pieces (see the definitions in [13], e.g.): the Weyl 2-form ${}^{(1)}R^{\alpha\beta}$, the traceless Ricci ${}^{(4)}R^{\alpha\beta}$ and the curvature scalar ${}^{(6)}R^{\alpha\beta}$ piece. Accordingly, there are three coupling constants b_1, b_4 and b_6 (with dimension of length square) in the theory. In tensor language, the Lagrangian (35) can be rewritten as

$$V = -\frac{1}{4\kappa_4}\left(\alpha R^2 + \beta \mathrm{Ric}_{\alpha\beta}\mathrm{Ric}^{\alpha\beta} + \gamma R^{\alpha\beta}{}_{\mu\nu}R_{\alpha\beta}{}^{\mu\nu}\right)\eta \tag{36}$$

in terms of the curvature tensor components. The new coupling constants are related to the original ones via $\alpha = (2b_1 - 3b_4 + b_6)/12$, $\beta = b_4 - b_1$, $\gamma = b_1/2$. Sometimes, only the scalar square R^2 and the Ricci square $\mathrm{Ric}^2_{\alpha\beta}$ terms are kept, whereas the total curvature quadratic term is "removed" from the Lagrangian by making use of the Euler topological invariant $d\Phi_P$ (as done in [21], for example). However, although the topological boundary term (17) does not affect the field equations, it *does change* (as any other boundary term in the Lagrangian) the definition of field momenta and, hence, the conserved quantities.

For the Lagrangian (35), we have

$$H_\alpha = \psi_\alpha, \qquad H_{\alpha\beta} = \frac{1}{2\kappa_4}\sum_{I=1,4,6} b_I {}^{*(I)}R_{\alpha\beta}. \tag{37}$$

The field equations yield for the Lagrange multiplier

$$\psi_\alpha = 2e^\beta \rfloor DH_{\alpha\beta} - (1/2)\vartheta_\alpha \wedge e_\mu \rfloor e_\nu \rfloor DH^{\mu\nu}. \tag{38}$$

Substituting all this into $\mathcal{F}_\alpha = -DH_\alpha + E_\alpha = 0$ we obtain the system of ten fourth-order gravitational field equations.

All the Einstein spaces, for which $\mathrm{Ric}_{\alpha\beta} = -\lambda g_{\alpha\beta}$, are solutions of the fourth-order system for any constant λ. As an example, we choose again the Kerr-AdS spacetime (18)-(20). Like for the Einstein theory, a regularization is required for asymptotically nonflat solutions. For this purpose, we again use the relocalization $V' = V + \alpha_0 d\Phi_P$, with the topological boundary term (17). For the models (35), this is equivalent to a redefinition of the constants b_1, b_4 and b_6. We choose $\alpha_0 = b_6/8\kappa_4$, and the direct computation for the Kerr-AdS solution then yields

$$\mathcal{Q}'[\partial_t] = \Lambda_0 \Omega M c^2, \qquad \mathcal{Q}'[\partial_\varphi] = -\Lambda_0 \Omega^2 M c a. \tag{39}$$

Here the constant $\Lambda_0 := (b_6 - b_1)\lambda/3$ is dimensionless. This qualitatively agrees (since $(b_6 - b_1)/3 = 4\alpha + \beta$) with the results of [21], where a noninvariant definition of the total energy in quadratic-curvature theories was proposed.

VI. CONCLUSION

We have presented a general definition of *invariant* conserved currents and charges for gravity models with diffeomorphism and local Lorentz symmetries. In our opinion, the advantage of our approach is that everything is directly derived from the Lagrangian. The latter fixes the physical laws that govern a system. On the contrary, it seems that the non-Noether currents do not have a direct physical meaning since they are in general unrelated to the Lagrangian (and hence to the physical laws encoded in it). For that reason we excluded them from our analysis. We believe that our invariant Noether currents are physically meaningful quantities because: They i) are well defined for every Lagrangian, ii) satisfy the reasonable condition that current vanishes $\mathcal{J} = 0$ for the trivial Lagrangian $V_0 = 0$, iii) lead to reasonable results when evaluated for known configurations. We have indeed verified that this approach yields satisfactory values of the total energy and angular momentum for asymptotically flat and asymptotically AdS solutions of the gravitational models in various dimensions, with various (minimal and nonminimal) coupling, and for various (linear and quadratic) Lagrangians. We thus generalize and improve the results [2, 3, 4, 5, 6, 7, 8]. We also confirm and strengthen the observation of [8] that all locally AdS spacetimes have zero invariant charges for any ξ both in Einstein's gravity and in quadratic-curvature gravity, which implies degeneracy of the vacuum in these models. Among the number of possible further developments (currently under investigation), we mention the interesting applications of this approach to supergravity, and to black hole thermodynamics along the lines of [2].

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[1] The volume n-form is $\eta := \vartheta^{\hat{0}} \wedge \cdots \wedge \vartheta^{\hat{n}}$, and $e_\alpha \rfloor \phi$ denotes the interior product of a form ϕ with the vectors of the frame e_α dual to the coframe basis ϑ^α. Latin and Greek indices label the coordinate and the local frame components, respectively (hats over an index denote individual components w.r.t. an anholonomic frame). For $p = 0, \ldots, n$, we define $\eta^{\alpha_1 \cdots \alpha_p} := {}^*(\vartheta^{\alpha_1} \wedge \cdots \wedge \vartheta^{\alpha_p})$ where * is the Hodge operator. For other notation see [13].

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